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COMPLETE CONTROLLABILITY CRITERIA FOR CLASSES OF MECHANICAL SYSTEMS WITH BOUNDED CONTROLS†

Ye. S. PYATNITSKII

Moscow

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A collection (class) of controllable dynamical systems, each described by Lagrange equations of the second kind, is considered. The class is defined by specifying the bounded domains in which the controls and generalized forces may take their respective values. The systems differ from one another both in the expression for the kinetic energy, which may be chosen at will from a set of positive-definite quadratic forms in the velocities (with coefficients, which depend on the coordinates), and in the generalized forces, which may vary within the same domain. Necessary and sufficient conditions are established for any such class to be completely controllable (i.e. for any system belonging to the class to be completely controllable). These conditions have an obvious physical meaning. In the case, for example, of robot manipulators, the conditions imply that a system is completely controllable if and only if the maximum values of the control torques exceed the corresponding torques of the other forces (weight, resistance, etc.) in absolute value. © 1997 Elsevier Science Ltd. All rights reserved.

The need to consider collections of systems (not just single systems) arises in control problems with incomplete information, particularly the control of technological systems whose parameters may vary arbitrarily over a wide range (the mass of the payload in manipulative systems, coefficients of resistance, the parameters of the environment, and so on). When designing models of biomechanical systems, it proves difficult to determine exact expressions for the kinetic energy and generalized forces. In that case one can speak only of domains in which the forces may vary, or assume the values of certain functionals (such as maxima or minima of various quantities) to be known. It is therefore natural to consider a whole collection (class) of controllable dynamical systems, each of which is described by Lagrange equations of the second kind. The class is defined by specifying the bounded domains in which the controls and generalized forces may take their respective values. The systems of the class may differ either in their expressions for the kinetic energy, which is chosen arbitrarily from a set of positive-definite quadratic forms in the velocities (with coefficients which depend on the coordinates), or in their generalized forces, which may vary within the same domain. By singling out a class of completely controllable Lagrange systems one can considerably simplify the solution of the controllability problem for a specific system when one's information about the active forces and parameters is incomplete.

Controllability conditions for classes guarantee that this property will be maintained when the parameters and the forces are varied, i.e. they are robust in nature. That this is so enables one to determine general regularities, free from the individual peculiarities of specific systems. The results extend to the case of non-natural systems in which the kinetic energy may be an arbitrary strictly convex function of the generalized velocities.

1. FORMULATION OF THE PROBLEM

We will consider the class of all controllable dynamical systems whose motion in independent generalized coordinates $q = \| q_i \|_{i=1}^{n}$ may be described by Lagrange equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i(q, \dot{q}, t) + \sum_k b_{ik}(q, \dot{q}, t) u_k(t) \tag{1.1}$$

$$i = 1, \ldots, n$$

Throughout, summation over the subscripts i, k, s will run from 1 to n.

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The kinetic energy T of each system of the class is chosen from a set of positive-definite quadratic forms

$$T = \tfrac{1}{2}\sum_{i,k} a_{ik}(q)\dot{q}_i\dot{q}_k, \quad \lambda_0\sum_i \dot{q}_i^2 \leqslant T \leqslant \lambda_1\sum_i \dot{q}_i^2 \tag{1.2}$$

$$\lambda_j = \text{const}, \qquad \lambda_j > 0$$

with continuously differentiable coefficients $a_{ik}(q)$.

The admissible controls $u(t)$ will be functions $u_i(t)$, summable over any finite interval and taking values in a bounded closed convex domain U, i.e.

$$u(t) \in U, \quad u(t) = \| u_i(t) \|_{i=1}^n, \quad \operatorname{conv} U = U \tag{1.3}$$

It is assumed that a bounded closed domain $D_0 \subset R^n$, within which the vector of generalized forces may vary, is given

$$Q(q,\dot{q},t) \in D_0 \subset R^n, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geqslant t_0 \tag{1.4}$$

This means that the functions $Q_i(q, \dot{q}, t)$ are not given; all that is known is the domain D_0 in which they may take values.

The elements of the matrix $B(q, \dot{q}, t) = \| b_{ik}(q, \dot{q}, t) \|_{i,k=1}^n$ are assumed to be uniformly bounded functions

$$| b_{ik}(q,\dot{q},t)| \leqslant b_0, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geqslant t_0 \tag{1.5}$$

Since the controls $u_i(t)$ may be summable functions, the solutions of Eqs (1.1) will be continuously differentiable functions $q(t)$ with absolutely continuous derivatives $\dot{q}(t)$. It is therefore assumed that the functions $Q_i(q, \dot{q}, t)$ and $b_{ik}(q, \dot{q}, t)$, considered on the set of functions $q(t)$, are summable functions of t over any finite interval.

The class of controllable Lagrange systems (1.1) is defined by the sets U and D_0 of numbers b_0, λ_0, λ_1. A specific system of the class is singled out by specifying T, the vector function $Q(q, \dot{q}, t)$ and the matrix $B(q, \dot{q}, t)$, subject to the above-mentioned functional and geometrical restrictions. Any such system is considered to be a member of the class.

The problem is to establish conditions for a class of Lagrange systems (1.1) to be completely controllable.

Definition 1.1. Following Kalman, we will say system (1.1) is completely controllable in the $2n$-dimensional phase space $\{q, \dot{q}\}$ for a set of bounded controls $u(t) \in U$ if, for any two points $s^0\{q^0, \dot{q}^0\}$ and $s^1\{q^1, \dot{q}^1\}$ of the state space, an admissible control and a finite time t_1 (each possibly different for each pair s^0, s^1) exist such that system (1.1) traverses the path s^0, s^1 in that time.

Definition 1.2. A class of controllable Lagrange systems (1.1) is said to be completely controllable on a set of bounded controls $u(t) \in U$ if each system in the class is completely controllable.

Complete controllability of a class of systems (1.1) implies that the boundary-value problem

$$q(t_0) = q^0, \quad \dot{q}(t_0) = \dot{q}^0, \quad q(t_1) = q^1, \quad \dot{q}(t_1) = \dot{q}^1 \tag{1.6}$$

will have a solution for any system of the class if the control $u(t) \in U$ and time t_1 are suitably chosen—they may be different for each system and each pair $s^0\{q^0, \dot{q}^0\}$ and $s^1\{q^1, \dot{q}^1\}$.

In this formulation of the problem the boundedness of the controls (1.3) is essential. In most of the literature on controllability, the functions $u(t)$ are not assumed to be bounded.

The desirability of considering a whole collection of systems rather than one specific system arises in various problems, particularly when analysing biomechanical systems [1, 2], when it is not possible to specify the systems T, Q and B rigorously. An analogous situation arises in robot control, when certain parameters (the payload mass, the coefficients of friction, the parameters of the environment, etc.) are not known and may vary over a range. In general, such uncertainty is typical of the description of technological systems whose parameters may vary arbitrarily within certain tolerances. Consideration of a class of systems also enables general regularities to be revealed since the individual properties of specific systems are not used.

The main part of this paper is devoted to investigating the complete controllability of systems of type (1.1) in which the right-hand sides are linear functions of the control. These systems are covariant relative to transformations of the generalized coordinates. Finally, we will consider classes of systems in which the generalized forces may depend non-linearly on the control $u(t)$, and systems that are not natural [3].

2. CONTROLLABILITY OF A SIMPLE CLASS

We will first consider a subclass of systems (1.1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}=u_i(t) \tag{2.1}$$

which is obtained from (1.1) by putting $Q \equiv 0$, $B = E$, where E is the identity matrix of order n. As to the controls $u(t)$, we will assume (for simplicity) that

$$u(t)\in U_1\{u(t): |u_i(t)| \leqslant h_i, \quad i=1,\dots,n\} \tag{2.2}$$

The criterion for complete controllability of systems of class (2.1) will be used in an essential way later, when we consider class (1.1) and further extensions.

Theorem 1. A class of Lagrange systems (2.1) is completely controllable on a set of bounded controls $u(t) \in U_1$ if and only if

$$h_0=\min_{1\leqslant i\leqslant n} h_i>0 \tag{2.3}$$

Proof. Necessity. Let us consider a system of class (2.1) in which $T = 0.5\lambda_0\Sigma_i\dot{q}_i^2$, $\lambda_0 > 0$. In that case $\lambda_0\ddot{q}_i = u_i(t)$. If k exists such that $h_k = 0$, it is not possible to go from a state s^0 in which $q_k^0 = \dot{q}_k^0 = 0$, along trajectories of the system, to a state s^1 in which $q_k^1 = -1$, $\dot{q}_k^1 = 0$, since $q_k(t) \equiv 0$. Therefore, when $h_0 = 0$, the class of systems (2.1) will contain an uncontrollable system, proving the necessity of condition (2.3).

Sufficiency. The proof will be carried out constructively, by explicit construction of an admissible control $u(t) \in U_1$ that steers an arbitrary system of type (2.1) from any initial state $s^0\{q^0, \dot{q}^0\}$ to an arbitrary given terminal state $s^1\{q^1, \dot{q}^1\}$ in a finite time. We will point out the key steps of the proof.

1. Consider the admissible control (of the same type as dry friction forces) [4]

$$u_i=-h_i \operatorname{sign}\dot{q}_i \tag{2.4}$$

The corresponding system (2.1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}=-h_i \operatorname{sign}\dot{q}_i \tag{2.5}$$

has a discontinuous right-hand side. The solutions of system (2.5) are defined [5, 6] as continuously differentiable functions $q(t)$, with absolutely continuous derivatives $\dot{q}(t)$, satisfying the differentiable inclusion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}\in F(\dot{q}), \quad F(\dot{q})=\|-h_i \operatorname{sgn}\dot{q}_i\|_{i=1}^{n} \tag{2.6}$$

almost everywhere, where

$$\begin{aligned}&\operatorname{sgn}\dot{q}_i=1, \quad \text{if} \quad \dot{q}_i>0; \quad \operatorname{sgn}\dot{q}_i=-1 \quad \text{if} \quad \dot{q}_i<0\\ &\operatorname{sgn}\dot{q}_i=\mu_i, \quad \mu_i\in[-1,+1] \quad \text{if} \quad \dot{q}_i=0\end{aligned} \tag{2.7}$$

In accordance with the definition of the solution of equations with discontinuous right-hand sides [5, 6], $q(t)$ is the limit of a sequence of Eqs (2.5) with continuous right-hand sides, the discontinuous functions being approximated in a special way by a sequence of continuous functions $\varphi_s(\dot{q}_i)$, $s = 1, 2, \ldots$ In the case of system (2.5) one considers, instead of sign $\dot{q}_i$, a sequence of continuous functions $\varphi_s(\dot{q}_i)$ that converge to the function sgn $\dot{q}_i$ defined in (2.7), which is upper semicontinuous with respect to inclusion [6].

The combination of all the partial limits of the sequence of solutions of (2.1) $\{q^s(t)\}$ for $u_i = \varphi_s(\dot{q}_i)$, taking all possible approximating sequences $\varphi_s(\dot{q}_i)$ into consideration, defines the set of all solutions of system (2.5). It turns out that this set is not empty and, in the case of system (2.5), is identical with the set of all solutions of the differential inclusion (2.6) with semicontinuous right-hand sides. The solution of system (2.5), thus defined, need not be unique.

Accordingly, considering system (2.5) further (and analogous kinds of systems with discontinuous right-hand sides), we will always have in mind the corresponding differential inclusion (2.6). By a theorem of Filippov [7, 8], for any solution $\bar{q}(t)$ of the inclusion (2.6), a summable function $\bar{u}(t) = \| \bar{u}_i(t) \|_{i=1}^n$

$$\bar{u}_i(t) = -h_i \operatorname{sgn} \dot{\bar{q}}_i(t), \quad \bar{u}_i(t) = \bar{\mu}_i(t) \in [-1, 1] \quad \text{if} \quad \dot{\bar{q}}_i(t) = 0 \tag{2.8}$$

exists such that, almost everywhere in the interval under consideration

$$\bar{u}_i(t) = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right)_{q = \bar{q}(t)} \tag{2.9}$$

This relationship enables us to define a summable function $\bar{\mu}_i(t)$ on the set where $\dot{\bar{q}}_i(t) = 0$.

The theorem on the variation of the kinetic energy T for any solution of the inclusion (2.6) states that

$$\dot{T} = -\sum_i h_i |\dot{q}_i| \tag{2.10}$$

Since T satisfies the inequalities in (1.2), it follows that $\Sigma_i |\dot{q}_i| \geq (Tn^{-1}\lambda_1^{-1})^{1/2}$ and, consequently, it follows from (2.10) that $\dot{T} \leq -h_0 \Sigma_i |\dot{q}_i| \leq -2\rho\sqrt{T}$, where $2\rho = h_0(n^{-1}\lambda_1^{-1})^{1/2} > 0$.

The differential inequality $\dot{T} < -2\rho\sqrt{T}$ will be valid for any solution of (2.6) or, what is the same, any solution of (2.5). Therefore, $d\sqrt{(T)}/dt \leq -\rho < 0$, implying the inequality $\sqrt{T} \leq \sqrt{(T_0)} - \rho(t - t_0)$, where T_0 is the initial value of the kinetic energy. Consequently, $T \equiv 0$ for $t \geq t_0 + \rho^{-1}\sqrt{(T_0)} = t_1$, and therefore $\dot{q}(t) \equiv 0$, $q(t) = \gamma =$ const.

Thus, any motion of system (2.5), allowing for possible non-uniqueness, will reach equilibrium in a finite time, the equilibrium state depending in general on the initial state and on the choice of solution from the set of all solutions of the inclusion (2.6).

If a state $s^0\{q^0, \dot{q}^0\}$ is selected and a motion is chosen arbitrarily from the set of all motions, system (2.5) will move along the corresponding trajectory $q^0(t)$ and, in a finite time t_1, reach some equilibrium state $M^0\{\gamma^0, 0\}$. Denote the function (2.9) corresponding to $q^0(t)$ and $u^0(t)$. Consequently, when $u = u^0(t)$ system (2.1), by virtue of (2.9), will transfer from $s^0\{q^0, \dot{q}^0\}$ to $M^0\{\gamma^0, 0\}$ in a finite time along the trajectory $q^0(t)$.

Similar reasoning shows that an admissible control $u = u^1(t)$ will steer system (2.1) from $s^1\{q^1, \dot{q}^1\}$ to some equilibrium state $M^1\{\gamma^1, 0\}$ in a finite time t_2 along a trajectory $q^1(t)$. The substitution $t \to t_2 - t$ transforms system (2.1) into a system of the same form with control $u = u^1(t_2 - t)$, since the left-hand side of the Lagrange equations (2.1) (in the steady case) is invariant under time inversion. Therefore, if $u(t) = u^1(t_2 - t)$, system (2.1) will transfer, in a finite time t_2, from $M^1\{\gamma^1, 0\}$ to $s^1\{q^1, \dot{q}^1\}$.

2. To complete the proof, it remains to show that an admissible control $u = u^2(t)$ exists that steers system (2.1) from $M^0\{\gamma^0, 0\}$ to $M^1\{\gamma^1, 0\}$ in a finite time. To do this, consider the vector function

$$q^2(t) = \gamma^0 + \tfrac{1}{2}(\gamma^1 - \gamma^0)(1 - \cos\omega t) \tag{2.11}$$

which relates the states M^0 and M^1 in a time $t_3 = \pi\omega^{-1}$. We will show that $q^2(t)$, defined for sufficiently small $\omega > 0$, will be a solution of (2.1) for some admissible control $u(t) \in U_1$. Substituting (2.11) into the left-hand side of (2.1), we get

$$u_i^2(t) = \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right]_{q = q^2(t)} = \omega^2 \psi_i(\cos\omega t, \sin\omega t) \tag{2.12}$$

where $\psi_i(x, y)$ are continuously differentiable functions. In that case $|\psi_i(\cos\omega t, \sin\omega t)| \leq r < \infty$, $t \geq 0$. Choosing the parameter ω so that $\omega^2 r < h_0$, we see that the function $u^2(t)$ in (2.12) will satisfy (2.12), i.e. it will be an admissible control. Consequently, the admissible control $u^2(t)$ will steer (2.1) from $M^0\{\gamma^0, 0\}$ to $M^1\{\gamma^1, 0\}$ in a time $t_3 = \pi\omega^{-1} < \infty$.

3. Finally, we deduce that $u^0(t) \in U_1$ steers system (2.1) from $s^0\{q^0, \dot{q}^0\}$ in a time $t_1 < \infty$ to $M^0\{\gamma^0, 0\}$, the control $u^2(t) \in U_1$ steers (2.1) from $M^0\{\gamma^0, 0\}$ to $M^1\{\gamma^1, 0\}$ in a time $t_3 = \pi\omega^{-1} < \infty$ and, finally, the control $u^1(t_2 - t) \subset U_1$ steers (2.1) from $M^1\{\gamma^1, 0\}$ to $s^1\{q^1, \dot{q}^1\}$ in a finite time t_2.

During the proof, no conditions other than (1.2) were imposed on the choice of system (2.1) (i.e. on the choice of the kinetic energy T). The previous arguments are therefore valid for any system of type (2.1), i.e. for the whole class of such systems. This proves the theorem.

Remark. The proof of Theorem 1 proceeded by actual construction of a control $u(t) \in U_1$ steering a system of type (2.1) from a state s^0 to a state s^1 in a finite time. The motion consists of a decelerating section, a motion with generalized velocities of small absolute value, and an accelerating section.

3. GENERAL CASE

We now consider the class of Lagrange systems (2.1), permitting the inclusion of generalized forces

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q(q, \dot{q}, t) + u \tag{3.1}$$

where $u(t)$ satisfies (1.3). System (1.1) becomes (3.1) if one puts $B(q, \dot{q}, t) \equiv E$. Let Φ denote the bounded closed set (defining the class (3.1)) in which the function $(-1)\,Q(q, \dot{q}, t)$ may take values, i.e.

$$(-1)\;Q(q, \dot{q}, t) \in \Phi, \quad \Phi \subset R^n, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geq t_0 \tag{3.2}$$

Theorem 2. The class of Lagrange systems (3.1) is completely controllable on a set of bounded controls $u(t) \in U$ if and only if positive ε exists such that

$$\Phi_\varepsilon \subset U, \qquad \operatorname{conv} U = U \tag{3.3}$$

where Φ_ε is a closed ε-neighbourhood of the set Φ in (3.2).

Proof. Necessity. If condition (3.3) is not satisfied, then for any $\varepsilon > 0$ a vector $a_\varepsilon \in \Phi_\varepsilon$ exists that does not belong to U. Since the point a_ε and the convex set U do not intersect, a hyperplane $(c_\varepsilon, x) + d_\varepsilon = 0$ exists separating a_ε and U. We may assume without loss of generality that c_ε is a unit vector ($\| c_\varepsilon \| = 1$). In that case $(c_\varepsilon, a_\varepsilon) + d_\varepsilon \leq 0$, $(c_\varepsilon, u) + d_\varepsilon > 0$ for all $u \in U$. Reasoning similarly for a sequence $\varepsilon_s \to 0$, we obtain sequences a^s, c^s ($\| c^s \| = 1$) and d^s. Extracting subsequences if necessary, we obtain $c^s \to c$, $d^s \to d$ and $a^s\ a \subset \Phi$. Consider the system of class (3.1) for which $T = 1/2\lambda_0\Sigma_i\dot{q}_i^2$, $Q = -a =$ const. Since $\lambda_0(c, \ddot{q}) = -a + u$ and by construction $(c, a) + d \leq 0$, $(c, u) + d \geq 0$, it follows that $\lambda_0(c, \ddot{q}) = (c, u) - (c, a) = (c, u) + d - (c, a) - d \geq 0$ for any $u \in U$. Therefore, it is not possible, moving along trajectories of this system from a state $s^0\{q^0, \dot{q}^0\}$ in which $(c, q^0) \geq 0$, $(c, \dot{q}^0) \geq 0$, to reach a state $s^1\{q^1, \dot{q}^1\}$, where $(c, q^1) < 0$, since in that case $(c, q(t)) \geq (c, q^0) + t(c, \dot{q}^0) \geq 0$ for all $t \geq 0$. Consequently, if condition (3.3) does not hold, the class of systems (3.1) will contain an uncontrollable system, proving the necessity of (3.3).

Sufficiency. Consider any system of the form (3.1). If $|w_i(t)| \leq 2^{-1}\varepsilon$, $\varepsilon > 0$, the control $u = -Q(q, \dot{q}, t) + w(t)$ will be admissible by condition (3.3), since in that case $u \in \Phi_{\varepsilon/2} \subset U$. With that choice of admissible control, system (3.1) becomes a system of type (2.1) with $u(t) = w(t)$, $|w_i(t)| \leq h_i$, $h_i = 2^{-1}\varepsilon$, $\varepsilon > 0$. By Theorem 1, this system will be completely controllable, so that the same is true of any system of class (3.1).

Condition (3.3) of Theorem 2 enables one to choose an admissible control that not only neutralizes the generalized forces $Q(q, \dot{q}, t)$ but also guarantees the presence of sufficient resources to ensure that the class of systems (3.1) is completely controllable. If the admissible control is chosen to have the form $u = -Q + w$, the class of systems (3.1) is transformed into class (2.1). When the set Φ in (3.2) is a parallelepiped and the set U has the form (2.2), condition (3.3) may be written as

$$\sup |Q_i(q, \dot{q}, t)| < h_i$$

These inequalities have an obvious physical meaning. For example, a robot manipulator with n degrees of freedom will belong to the class of completely controllable Lagrange systems if the largest possible values h_i of the controlling torques of the drives are greater (in absolute value) than the upper limit of the corresponding torques of the forces of gravity, resistance, etc.

It should be noted that the condition of complete controllability for (3.3) establishes a relationship only among the generalized forces and controlling forces. The parameters of the essentially non-linear left-hand side of the equations of dynamics defined by the Euler–Lagrange operator have no effect whatever on the complete controllability condition, if T satisfies (1.2). This means that the complete controllability property for a class of mechanical systems depends not on the structure of the system but exclusively on the active and controlling forces.

Theorem 2 also implies an important conclusion concerning the minimum possible number of independent controls. Namely: if a class of mechanical systems (3.1) with n degrees of freedom is completely controllable, the number of independent controls $u_i(t)$ must be at least n.

We will now analyse the class of systems (1.1). Let us consider the subclass of these systems defined by the inequality

$$|\det B(q,\dot{q},t)| \geq \Delta > 0, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geq t_0 \tag{3.4}$$

where $\Delta > 0$ is a fixed positive number. The number Δ, together with D_0, U, b_0, λ_0 and λ_1, define the subclass uniquely.

To understand the meaning of condition (3.4), consider the system of class (1.1) with $T = 1/2\lambda_0 \Sigma_i \dot{q}_i^2$, $Q = P = \text{const}$, $P \in D_0$, $b_{1k}(q, \dot{q}, t) = 0$ for all $1 \leqslant k \leqslant n$. In that case $\det B = 0$ and $\ddot{q}_1 = p_1$. Putting $p_t \geqslant 0$ to fix our ideas, we see that moving along the trajectories of the system from a state s^0 in which $q_1^0 = \dot{q}_1^0 = 0$, it is not possible to reach a state s^1 in which $q_1^1 \leqslant 0$, since $q_1 \geqslant 0$.

Thus, as this example shows, condition (3.4) enables us to single out a subclass of systems (1.1) that does not contain uncontrollable systems. The question of whether condition (3.4) is necessary in the general case is still open.

Theorem 3. The subclass of systems (1.1) defined by condition (3.4) is completely controllable if and only if a positive number exists such that

$$\Phi_{1\varepsilon} \subset U \tag{3.5}$$

where $\Phi_{1\varepsilon}$ is the closed ε-neighbourhood of the bounded set Φ_1 within which the vector function $(-1)B^{-1}(q, \dot{q}, t)Q$ is allowed to vary, where $Q \subset D_0$, $q \in R^n$, $\dot{q} \in R^n$, $t \geqslant t_0$.

Proof. Necessity. If condition (3.5) does not hold, then for any $\varepsilon > 0$ a vector $a_\varepsilon \in \Phi_{1\varepsilon}$ exists that is not in U. By the separation theorem for convex sets, a hyperplane $(c_\varepsilon, u) + d_\varepsilon = 0$ exists separating a_ε and U, where $\| c_\varepsilon \| = 1$. A sequence $\varepsilon_s \to 0$ induces sequences a^s, $c^s(\| c^s \| = 1)$ and d^s. Extracting subsequences if necessary, we obtain $c^s \to c$, $a^s \to a \in \Phi_1$, $d^s \to d$, where $(c, a) + d \leqslant 0$ and $(c, u) + d \geqslant 0$ for all $u \in U$.

Consider the system of class (1.1) for which $T = 1/2\lambda_0\Sigma_i\dot{q}_1^2$, $B = \text{const}$, $B^{-1}Q = -a$, $a \subset \Phi_1$. It follows from the equations of motion of this system that $\lambda_0\ddot{q} = Bu - Ba$; hence

$$\lambda_0(c, B^{-1}\ddot{q}) = (c,u) + d - (c,a) - d \geq 0.$$

This means that

$$(c, B^{-1}q(t)) \geq (c, B^{-1}q^0) + t(c, B^{-1}\dot{q}^0) \geq 0.$$

It is therefore impossible, moving along trajectories of this system from a state $s^0\{q^0, \dot{q}^0\}$, where $(c, B^{-1}q^0) \geqslant 0$, $(c, B^{-1}\dot{q}^0) > 0$, to reach a state $s^1\{q^1, \dot{q}^1\}$, where $(c, B^{-1}q^1) < 0$. Consequently, if condition (3.5) is not valid, the subclass of systems (1.1) defined by inequality (3.4) will contain an uncontrollable system, proving that condition (3.5) is indeed necessary.

Sufficiency. Define a control by

$$u = -B^{-1}(q,\dot{q},t)Q(q,\dot{q},t) + B^{-1}(q,\dot{q},t)w$$

If $|w_1(t)| \leqslant 2^{-1}n^{-1}c_0^{-1}\varepsilon = \mu$, where $c_0 \geqslant |c_{ik}(q, \dot{q}, t)|$, and $C(q, \dot{q}, t) = B^{-1}(q, \dot{q}, t)$, this control will be admissible because $u \subset \Phi_{1\varepsilon/2} \subset U$. That the elements of the matrix $C(q, \dot{q}, t)$ are uniformly bounded follows from (3.4) and (1.5). With the above choice of a control, systems of type (1.1) satisfying (3.4) become systems of type (2.1) for $u_i(t) = w_i(t)$, where $|w_1(t)| \leqslant \mu$, $\mu > 0$. By Theorem 1, all these systems will be completely controllable, hence so is class (1.1).

4. NON-LINEAR DEPENDENCE OF THE FORCES ON THE CONTROLS

We have been considering Lagrange systems in which the right-hand sides of the equations of motion are linear in the controls. We will now consider the general class of controllable Lagrange systems

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q(q, \dot{q}, t, u), \quad u \in U, \quad \operatorname{conv} U = U \tag{4.1}$$

We will define a class of systems (4.1) which we call the class of solvable systems.

A class (4.1) is said to be solvable if a positive ε exists such that the equations

$$Q(q, \dot{q}, t, u) = w, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geq t_0 \tag{4.2}$$

have an admissible solution

$$u = u_0(q, \dot{q}, t, w) \subset U, \quad q \in R^n, \quad \dot{q} \in R^n, \quad t \geq t_0 \tag{4.3}$$

for all w in the sphere $\|w\| \leqslant \varepsilon$.

All the classes of systems considered hitherto, including the subclass (1.1) and (3.4), belonged to the set of solvable systems.

Theorem 4. The class of solvable Lagrange systems (4.1) is completely controllable on a set of bounded controls $u \in U$.

The truth of this theorem follows directly from Theorem 1. Indeed, if one takes a function $u_0(q, \dot{q}, t, w)$ as in (4.3) as an admissible control in (4.1), the class (4.1) becomes the class (2.1), which is completely controllable for $\|w\| \leqslant \varepsilon$.

Note that the complete controllability concept is strongly related to the issue of whether some motion $q^*(t)$ is realizable, i.e. the issue of the existence of an admissible control satisfying the condition

$$Q(t, \dot{q}^*, t, u) = \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q}\right]_{q=q^*(t)}$$

This equation is essentially an equation of type (4.2) for $u(t)$.

Note that the analytic solvability conditions (4.3) may be obtained using the implicit function theorem, if the equation $Q(q, \dot{q}, t, u) = 0$ has a solution $\bar{u}(q, \dot{q}, t)$ that varies within a closed domain $\bar{U} \subset \operatorname{int} U$.

5. THE CONTROLLABILITY OF CLASSES OF NON-NATURAL SYSTEMS

In this section the results obtained above will be extended to classes of systems that are not natural. The kinetic energy T of such systems may be an arbitrary strictly convex function of the generalized velocities $\dot{q}_r$, not necessarily representable in the form (1.2).

In the previous presentation Theorem 1 was of central importance, as the complete controllability conditions formulated in Theorems 2–4 enabled us to reduce the problem to systems of type (2.1). With this in mind, we will only establish an analogue of Theorem 1 for non-natural systems. In the proof of Theorem 1 we used the theorem on the variation of the kinetic energy $T = \Sigma_i Q_i \dot{q}_i$. Since the analogue of this theorem for non-natural systems is the condition

$$\frac{d}{dt}\left(\Sigma_i \dot q_i \frac{\partial T}{\partial \dot q_i} - T\right) = \Sigma_i Q_i \dot q_i$$

one naturally changes to canonical variables $\{q, p\}$, defining the generalized momenta p in terms of the Legendre transformation $p_i = \partial T/\partial \dot q_i$. As we know, the Legendre transformation defines a diffeomorphism of the spaces $\{q, \dot q\}$ and $\{q, p\}$ if $T(q, \dot q)$ is a strictly convex function of $\dot q$ such that

$$\Sigma_i(\partial T / \partial \dot q_i)^2 \to \infty \text{ as } \Sigma_i \dot q_i^2 \to \infty$$

In that case [9] the Hamiltonian

$$H(q,p) = \max_{\dot q \in R^n}[(p, \dot q) - T(q, \dot q)]$$

will be strictly convex in p, with a unique minimum point $p = 0$ ($H(q, 0) = 0$), and the equations of motion of systems of class (2.1) will be

$$\dot q_i = \partial H / \partial p_i, \quad \dot p_i = -\partial H / \partial q_i + u_i(t), \quad u \in U_1 \tag{5.1}$$

(the set U_1 was defined in (2.2)). When generalized forces are present, the right-hand sides of Eqs (5.1) for the momenta p will contain terms $Q_i(q, p, t)$. Let us consider a class of controllable systems (5.1) in which $H(q, p)$ may be an arbitrary strictly convex function of the momenta p of class c^2 satisfying the inequalities

$$\varphi_1(\|p\|) \le H(q, p) \le \varphi_2(\|p\|)$$

$$\varphi_3(\|p\|) \le \Sigma_i(\partial H / \partial p_i)^2, \quad p \in R^n, \quad q \in R^n \tag{5.2}$$

where $\varphi_s(\xi)$ ($\varphi_s(0) = 0$) are continuous strictly increasing functions with infinitely large lower limit [10].

The only stationary part of the Hamiltonians $H(q, p)$ that we have admitted for consideration is $p = 0$. The functions $Q_s(\xi)$ in (5.2) determine the class of systems (5.1) under consideration and are assumed to be given. In the natural systems considered above, $Q_s(\xi) = a_s\xi^2$, since $H(q, p) = 0.5\Sigma_{ik}\alpha_{ik}(q)p_i p_k$.

Under these assumptions the following assertion holds.

Theorem 5. The class of systems (5.1) just described is completely controllable if and only if condition (2.3) holds.

Proof. Necessity. If k exists for which $h_k = 0$, the class (5.1) will contain uncontrollable systems. Indeed, consider $H = \Sigma_i H_i(p_i)$. In this case $\dot p_k = 0$. Therefore, it is not possible to go from a state $s^0\{q^0, p^0\}$ to a state $s^1\{q^1, p^1\}$ with $p_k^1 \ne p_k^0$.

Sufficiency. Choose any two points $s^0\{q^0, p^0\}$ and $s^1\{q^1, p^1\}$ of the phase space $\{q, p\}$ and consider the control $u_i = -h_0 \operatorname{sign}(\partial H/\partial p_i)$. The corresponding system (5.1) will be

$$\dot q_i = \partial H / \partial p_i, \quad \dot p_i = -\partial H / \partial q_i - h_0 \operatorname{sign}(\partial H / \partial p_i) \tag{5.3}$$

The total derivative of H along trajectories of (5.3), i.e. along trajectories of the differential inclusion corresponding to (5.3)

$$\dot q = \partial H / \partial p, \quad \dot p + \partial H / \partial q \in -h_0 \|\operatorname{sgn} \partial H / \partial p_i\|_{i=1}^n$$

(the function sgn η was defined in (2.6)), will be

$$\dot H = -h_0 \Sigma_i |\partial H / \partial p_i|$$

and by (5.2) we obtain the estimate $\dot H \leqslant -h_0(n^{-1}\varphi_3(\| p \|))^{1/2} < 0$ for $p \ne 0$. Inequalities (5.2) and the strict monotonicity of the functions $\varphi_s(\| p \|)$ imply an estimate $\| p \| \geqslant \psi_2(H)$, which leads to the inequality

$$\dot{H} \leq -h_0(n^{-1}\varphi_3(\psi_2(H)))^{1/2}$$

Hence it follows that the manifold $H = 0$ is asymptotically stable. It follows from the last inequality and from the estimates (5.2), which are uniform in $q \in R^n$, that for any $\varepsilon > 0$ and $\Delta > 0$ $t(\varepsilon, \Delta)$ exists such that $\| p(t) \| \leqslant \varepsilon$ for all $\| p \| \leqslant \varepsilon$, provided that $\| p_0 \| \leqslant \Delta$. In the sphere $\| p \| \leqslant \varepsilon$, where $\varepsilon > 0$ is sufficiently small, the function $H(q, p)$, which is strictly convex in p, admits of a representation

$$H(q, p) = 0{,}5\Sigma_{ik}c_{ik}(q)p_ip_k + \psi(q, p),\ c_{ik} = (\partial^2 H / \partial p_i \partial p_k)_{p=0} \tag{5.4}$$

where $| \psi(q, p) | \leqslant \rho \| p \|^2$ for sufficiently small $\rho > 0$. Since $H(q, p)$ is strictly convex in p, it follows that $\Sigma_{ik}c_{ik}(q)p_ip_k \geqslant \mu_1 \Sigma_i p_i^2$, $\mu_1 = \text{const}$, $\mu_1 > 0$. Hence, in the sphere $\| p \| \leqslant \varepsilon$, where $\varepsilon > 0$ is sufficiently small, we have $H(q, p) \geqslant \mu \Sigma_i p_i^2$, $\mu = \text{const}$, $\mu > 0$. Consequently, $| p_i | \leqslant (\mu^{-1}H)^{1/2}$ in this sphere and the derivative $\dot{H} = -h_0\Sigma_i | \partial H/\partial p_i |$ satisfies the estimate

$$\dot{H} \leq -h_0(\mu^{-1}H)^{1/2} = -2\beta(H)^{1/2},\ \beta > 0 \tag{5.5}$$

Indeed, it follows from the convexity of $H(q, p)$ as a function of p that

$$H(q, p) \leq \Sigma_i p_i \partial H / \partial p_i \leq \Sigma_i |p_i|\, |\partial H / \partial p_i| \leq (\mu^{-1}H)^{1/2} \Sigma_i |\partial H / \partial p_i|$$

whence it follows that $(\mu H)^{1/2} \leqslant \Sigma_i |\partial H/\partial p_i|$, implying the validity of (5.5).

Therefore, $d/dt H^{1/2} \leqslant -\beta < 0$, which implies $H^{1/2} \leqslant H_0^{1/2} - \beta(t - t_0)$. Taking (5.2) into account, we obtain $\varphi_1^{1/2}(\| p \|) \leqslant \varphi_2^{1/2}(\| p \|) - \beta(t - t_0)$, which directly implies the identity $p(t) \equiv 0$ for $t \geqslant t_0 + \beta^{-1}\varphi_2^{1/2}(\| p \|)$. Since $\partial H/\partial p = 0$ for $p = 0$, it follows that $\dot{q}(t) \equiv 0$ for $t \geqslant t_0 + \beta^{-1}\varphi_2^{1/2}(\| p_0 \|)$ and $q(t) = \gamma$. This may also be established using a theorem due to Rumyantsev [10], in combination with a theorem from [11].

Thus, system (5.3) will take a finite time to move along a trajectory $\{q^0(t), p^0(t)\}$, which may be chosen arbitrarily from the set of all solutions of (5.3), from a state $s^0\{q^0, p^0\}$ to an equilibrium state $M^0\{\gamma^0, 0\}$. As in the proof of Theorem 1, one can define on the motion $\{q^0(t), p^0(t)\}$ a summable function $u^0(t) \in U_1$ which steers (5.1) from s^0 to M^0.

In exactly the same way one shows that the system

$$\dot{q}_i = -\partial H / \partial p_i,\ \dot{p}_i = \partial H / \partial p_i - h_0 \operatorname{sign}(\partial H / \partial p_i)$$

obtained from (5.1) by time inversion, will proceed in a finite time t_2 from s^1 $\{q^1, p^1\}$ to $M^1\{\gamma^1, 0\}$ along a trajectory $\{q^1(t), p^1(t)\}$ defined by an admissible control $u^2(t)$. Inverting time in this latter system, we conclude that system (5.1) will transfer from M^1 to s^1 under the control $u = u(t_2 - t)$.

We claim that a control $u^3(t) \in U$ exists that steers (5.1) from $M^0\{\gamma^0, 0\}$ to $M^1\{\gamma^1, 0\}$ in a finite time. To prove this, consider the function (2.11), which joins the points M^0 and M^1 in $\{q, \dot{q}\}$ space in time $\pi\omega^{-1}$. Since the Legendre transformation defines a diffeomorphism of $\{q, \dot{q}\}$ and $\{q, p\}$ spaces, the value $\dot{q} = 0$ corresponds to the single point $p = 0$, since $p = 0$ is the unique point at which $\partial H/\partial p = 0$. Consequently, the function (2.11) will also join M^0 and M^1 in $\{q, p\}$ space.

We claim that this trajectory may be realized in (5.1) using an admissible control, and moreover in such a way that $\| p(t) \| \leqslant \varepsilon$. For sufficiently small $\| q \|$, corresponding to $\| p \| \leqslant \varepsilon$, proceeding as for (5.4), we obtain

$$T = \tfrac{1}{2}\Sigma_{i,k}a_{ik}(q)\dot{q}_i\dot{q}_k + \psi_1(q, \dot{q})$$

and so $p_i = \partial T/\partial \dot{q}_i = \Sigma_s q_{is}(q, \dot{q})\dot{q}_s$, which defines functions $p_i(t) = \omega\, \psi_i(\sin \omega t, \cos \omega t)$ and $\dot{p}_i(t) = \omega^2\varphi_i(\sin \omega t, \cos \omega t)$ with continuous functions $\varphi_i(x, y)$ and $\psi_i(x, y)$ on the motion (2.11). For $\| p \| \leqslant \varepsilon$, it follows from (5.4) that on the motion (2.11) $\partial H/\partial p_i = \omega^2\Phi_i(\sin \omega t, \cos \omega t)$. Therefore, putting $u_i^3(t) = \dot{p}_i(t) + \partial H/\partial q_i = \omega^2 R(\sin \omega t, \cos \omega t)$, we conclude that $u^3(t) \in U_1$ for sufficiently small ω. Choosing ω so that $u^3(t) \in U_1$ and $\| p(t) \| \leqslant \varepsilon$, we see that $u^3(t)$ will steer system (5.1) from M^0 to M^1 in a finite time. Since the specific form of the Hamiltonian $H(q, p)$ was not used in the proof, all the arguments remain valid for all systems of class (5.1), where the Hamiltonian satisfies (5.2). This completes the proof of Theorem 5.

Analogues of Theorems 2, 3, and 4 for non-natural systems may be established using Theorem 5—the proofs are verbatim repetitions of those given here.

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